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Gaussian binomials and the number of sublattices

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The purpose of this short communication is to make some observations on the connections between various existing formulas of counting the number of sublattices of a fixed index in an n-dimensional lattice and their connection with the Gaussian binomials.

1. Existing formulas

There are various ways of determining the number of sublattices of a fixed index in a lattice, they can be found in Cassels (1971), Baake (1997) and Gruber (1997). To determine the number $f_n(m)$ (notation as in Baake, 1997) of sublattices of index m in an n-dimensional lattice is the same as to determine the number of subgroups of index m in a free abelian group of rank n. A detailed discussion of this problem was included in Baake (1997), where a formula to compute $f_n(m)$,

$$f_n(m) = \sum_{d_1 d_2 \ldots d_n = m} d_1^0 d_2^1 \ldots d_n^{n-1}, \tag{1}$$

and a formula to express the generating function $F_n(s)$ of $f_n(m)$ as a Dirichlet series [$\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ is the Riemann zeta function],

$$F_n(s) = \zeta(s)\zeta(s-1)\ldots\zeta(s-n+1), \tag{2}$$

are provided. These formulas imply the following recursion relation:

$$f_n(m) = \sum_{d|m} d f_{n-1}(d).$$

One can also use the results on pp. 11–13 from Cassels (1971) to show that $f_n(m)$ is equal to the number of $n \times n$ matrices (r_{ij}) with integer entries satisfying the conditions (lower triangular matrices)

$$\begin{aligned} r_{ij} &= 0, \quad 1 \le i < j \le n; \\ r_{ii} > r_{ij} &\ge 0, \quad 1 \le j < i \le n; \\ r_{11} r_{22} \ldots r_{nn} &= m. \end{aligned} \tag{3}$$

Gruber (1997) proved that, if $m = p_1^{r_1} \ldots p_k^{r_k}$ is the prime factorization of m, then $f_n(m)$ can also be computed by the following formula:

$$f_n(m) = \prod_{i=1}^{k} \prod_{j=1}^{r_i} \frac{p_i^{n+j-1} - 1}{p_i^j - 1} = \prod_{i=1}^{k} \prod_{j=1}^{n-1} \frac{p_i^{r_i+j} - 1}{p_i^j - 1}. \tag{4}$$

Although all these methods were mentioned in Gruber (1997), the connections among these existing methods have not been adequately explained. In the next section, we will make some observations on the connections among these methods as well as the connection with Gaussian binomials.

2. Observations

First, we observe that formula (1) can be derived from Cassels's result by noting that, for an integer matrix satisfying condition (3), there are r_{ii} choices for each of the elements below r_{ii} at the ith column, and therefore the number of these matrices for each decomposition $m = r_{11} r_{22} \ldots r_{nn}$ is $r_{11}^{n-1} r_{22}^{n-2} \ldots r_{nn}^0$. Summing over all decompositions, one gets equation (1).

Then we observe that formula (4) can be derived from formula (2) by using the Gaussian binomials. Recall [Jantzen (1996), chapter 0 or the online Wikipedia] that, for integers $m, k \ge 0$, the Gaussian binomials (q-binomial coefficients) are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[m-k]_q! [k]_q!}, \tag{5}$$

where

$$[m]_q = \frac{1 - q^m}{1 - q}, \quad [m]_q! = [1]_q [2]_q \ldots [m]_q. \tag{6}$$

These binomials satisfy

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \begin{bmatrix} m \\ m-k \end{bmatrix}_q. \tag{7}$$

By using Gaussian binomials, one has the following formula:

$$\prod_{k=0}^{n-1} \frac{1}{1 - q^k t} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k. \tag{8}$$

Now since

$$\begin{aligned} F_n(s) = \prod_{i=0}^{n-1} \zeta(s-i) &= \prod_p \prod_{k=0}^{n-1} \frac{1}{1 - p^{-s+k}} \\ &= \prod_p \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_p p^{-sk} \\ &= \sum_{m=1}^{\infty} \frac{\prod_{i=1}^{r_m} \begin{bmatrix} n+k_i-1 \\ k_i \end{bmatrix}_{p_i}}{m^s}, \end{aligned}$$

where $m = p_1^{k_1} \ldots p_{r_m}^{k_{r_m}}$, we obtain the first formula in (4) and, applying (7), we obtain the second formula in (4).

3. Concluding remarks

There exist two approaches to the problem of counting the number of sublattices of a fixed index in the literature. The most detailed discussion is provided by Baake (1997). Alternatively, one can derive (1) from Cassels (1971), and then prove (2) by using arguments similar to those of Baake (1997). The connection between the number of sublattices of fixed indices and the Gaussian binomials is provided by the two product formulas in Gruber (1997).

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